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COMPLEX VERSUS DIFFERENTIABLE CLASSIFICATION OF ALGEBRAIC SURFACES

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We announce some results concerning the diffeomorphism classification of algebraic surfaces. A coefficient of Donaldson's invariants for a simply connected elliptic surface is calculated. This calculation implies a finiteness result for the moduli space of all complex structures on a fixed diffeomorphism class which has an algebraic surface as representative, as well as restrictions on the possible self-diffeomorphisms of certain algebraic surfaces.

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4-manifold
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Donaldson invariant
stable vector bundle
anti-self-dual connection

In this paper we state results which answer some of the questions and resolve some of the conjectures made in [9]. These results should be viewed as generalizations to general simply connected elliptic surfaces and complete intersections of the results in [7, 8] for Dolgachev surfaces. The technique of proof is to employ Donaldson's polynomial invariants $\gamma_c(M) \in \mathrm{Sym}^{d(c)}(H_2(M)^*)$ for smooth 4-manifolds M (see [3]). These polynomial invariants are constructed from the moduli space of anti-self-dual connections on the principal SU(2)-bundle P_c over M with $c_2(P_c) = c$, and are defined for all c sufficiently large. Furthermore, the degree $d(c)$ is equal to $4c - 3(b_2^+(M)+1)/2$. When the 4-manifold is an algebraic surface, these moduli spaces can be identified with the moduli spaces of stable rank-2 vector bundles over the surface (see [1, 2]).

To state our main result, we recall the notion of deformation equivalence of complex manifolds. Let M be a C^∞-manifold of dimension $2n$. Let $T^{(0,1)}$ be a C^∞ sub-bundle of the complexified tangent bundle $T_{\mathbb{C}}(M) \cong T(M) \otimes \mathbb{C}$ of M satisfying $T^{(0,1)} \oplus T^{(1,0)} \cong T_{\mathbb{C}}M$, where $T^{(1,0)}$ is the the complex conjugate of $T^{(0,1)}$. The subspace $T^{(0,1)}$ determines a complex structure on M if and only if the space of C^∞-sections of $T^{(0,1)}$ is closed under the operation of Lie bracket on vector fields. We say that

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two compact complex manifolds M_0 and M_1 are *deformation equivalent* if there is a diffeomorphism $f: M_0 \to M_1$ and a continuous path $T_t^{(0,1)}$ of subspaces determining complex structures on the C^∞-manifold underlying M_0, $0 \leq t \leq 1$, such that $T_0^{(0,1)}$ determines the given structure on M_0 and $T_1^{(0,1)}$ determines the complex structure pulled back from M_1 by f. Deformation equivalence is clearly an equivalence relation on complex structures, and tautologically, two deformation equivalent complex manifolds are diffeomorphic. In fact, a complex structure determines an orientation on the manifold, and two deformation equivalent complex manifolds are orientation-preserving diffeomorphic.

Our first main result is the following theorem.

Theorem 1. *The natural map*

$$\left\{\begin{matrix} \textit{algebraic surfaces modulo} \\ \textit{deformation equivalence} \end{matrix}\right\} \to \left\{\begin{matrix} \textit{smooth 4-manifolds modulo} \\ \textit{orientation-preserving diffeomorphism} \end{matrix}\right\}$$

is finite-to-one.

The proof of Theorem 1 involves the classification of algebraic surfaces. In fact, a rather straightforward reduction shows that it is enough to prove Theorem 1 for the subset of algebraic surfaces consisting of (possibly blown up) simply connected elliptic surfaces. Let S be such a surface. Then the deformation equivalence class of S is specified by four integers p_g, p_1, p_2, r. Here $p_g = p_g(S)$ is the geometric genus ($b_2^+(S) = 2p_g + 1$), p_1 and p_2 are two relatively prime integers ≥ 1 which are the multiplicities of the multiple fibers, and $r \geq 0$ is the number of blow ups of the minimal model $S_{\min}$ of S in the natural map $S \to S_{\min}$; equivalently, $r = -c_1^2(S) = \chi(S) - (1+p_g)/12$. If $p_g = 0$ and at least one of the $p_i = 1$, then S is a rational surface, and hence is diffeomorphic to $\mathbb{C}P^2 \# (9+r)\overline{\mathbb{C}P^2}$. If $p_g = 0$ and both p_1 and p_2 are greater than 1, then S is a *Dolgachev surface*; these surfaces are considered in [7, 8]. Given the results of [7], Theorem 1 is a consequence of the following theorem.

Theorem 2. *Suppose that S and S′ are simply connected blown up elliptic surfaces, blown up r and r′ times respectively from their minimal models. Suppose that* $p_g(S) > 0$. *Let* p_1 *and* p_2 *be the multiplicities of the multiple fibers of S, and let* p_1' *and* p_2' *be those of S′. If S is diffeomorphic to S′, then* $r = r'$, $p_g(S) = p_g(S')$ *and* $p_1 p_2 = p_1' p_2'$.

For any complex surface S we denote by K_S the canonical class of S and by κ_S the primitive integral class in $H^2(S; \mathbb{Z})$, which is a positive rational multiple of K_S. Recall from [6, Theorem 5] that if S is a surface with $p_g > 0$ and if S is a minimal simply connected elliptic surface or a complete intersection, then Donaldson's polynominal invariants $\gamma_c(S) \in \mathrm{Sym}^{d(c)}(H_2(S)^*)$ are in fact polynomials in κ_S and the intersection form $q_S \in H^2(S) \otimes H^2(S)$:

$$\gamma_c(S) = f_{S,c}(\kappa_S, q_S).$$

This is a consequence of the fact that the monodromy representation for the universal family of such surfaces has a large image in the component group of the diffeomorphism group of S. The case of Theorem 2 for minimal elliptic surfaces is deduced from the following computation of the leading coefficient of $f_{S,c}$ in the case when S is a simply connected elliptic surface.

Proposition 3. *Let S be a simply connected minimal elliptic surface with $p_g > 0$ and with multiple fibers of multiplicities p_1 and p_2. Set $d = d(c) = 4c - 3(p_g(S)+1)$, and write*

$$\gamma_c(S) = \sum_{i=0}^{[d/2]} a_i q_S^i \kappa_S^{d-2i}.$$

Assume that $c > 2(p_g+1)$, and set $n = 2c - 2p_g - 1$. Then $a_i = 0$ for $i > n$, and

$$a_n = \frac{d!}{2^n n!}(p_1 p_2)^{p_g}.$$

Proposition 3 is a consequence of the analysis of stable rank-2 vector bundles over simply connected elliptic surfaces given in [5].

To extend this result to one that implies Theorem 2 for blown up elliptic surfaces requires an understanding of the relationship between $\gamma_c(M)$ and $\gamma_c(M \# \overline{\mathbb{C}P^2})$ for general simply connected 4-manifolds M. Once we have this relationship and Proposition 3, Theorem 2 follows from a purely algebraic analysis of the nature of the invariants.

Here is a simple topological consequence of Theorem 2 and of Freedman's classification [4] of simply connected, topological 4-manifolds.

Corollary 4. (a) *For every $k \geq 1$ and $N \geq 10k - 1$, there are infinitely many pairwise distinct smooth structures on the topological 4-manifold $(2k-1)\mathbb{C}P^2 \# N\overline{\mathbb{C}P^2}$.*

(b) *Let X be the $K3$ surface. For every $l \geq 1$, the manifold $lX \# (l-1)S^2 \times S^2$ admits infinitely many pairwise distinct smooth structures.*

There are also results concerning self-diffeomorphisms of blow ups of certain surfaces. Let S be a surface of non-negative Kodaira dimension. Suppose that $\pi : S \to S_{\min}$ is the blow down of S to its minimal model and that (for simplicity) S is obtained from $S_{\min}$ by blowing up distinct points $x_1, \ldots, x_r$. Let $\pi^{-1}(x_i) = E_i$. If D is a divisor class on S, then we denote by $[D]$ the associated element of $H^2(S; \mathbb{Z})$. Thus, from the point of view of algebraic geometry, $H^2(S; \mathbb{Z})$ has distinguished classes $[E_i]$ and $\pi^*[K_{S_{\min}}]$. Using the result that simply connected elliptic surfaces and complete intersections have large monodromy groups [6, Proposition 4] and the calculation of the relationship between $\gamma_c(M)$ and $\gamma_c(M \# \overline{\mathbb{C}P^2})$ alluded to above, we have the following theorem.

Theorem 5. (a) *Let S be a blown up simply connected elliptic surface or a blown up complete intersection with* $p_g > 0$. *Let* $f: S \to S$ *be a diffeomorphism. Then f permutes the classes* $\{\pm[E_i]\}$. *Equivalently, f preserves the orthogonal direct sum decomposition*

$$H^2(S; \mathbb{Z}) \cong \pi^* H^2(S_{\min}; \mathbb{Z}) \oplus \bigoplus_i \mathbb{Z}[E_i].$$

(b) *Suppose that S is a blown up simply connected elliptic surface and that* $p_g(S) \neq 1$. *Then* $f^* \pi^*[K_{S_{\min}}] = \pm \pi^*[K_{S_{\min}}]$.

(c) *Suppose that S is a blown up complete intersection surface with* $p_g(S) \equiv 0 \pmod 2$ *and that* $p_g(S) > 0$. *Then* $f^* \pi^*[K_{S_{\min}}] = \pm \pi^*[K_{S_{\min}}]$.

Note. It is likely that the techniques of proof can be pushed further to handle (b) in the case when $p_g = 1$ as well.

According to [3], if a simply connected algebraic surface S is a connected sum $S \cong A \# B$, then one of the factors has a negative definite intersection form. In particular, if the intersection form of S is even, then one of the factors is a homotopy sphere. There is an extension of Theorem 5(a) which gives a refinement of this result.

Theorem 6. *Let S be a blown up simply connected elliptic surface or a blown up complete intersection with* $p_g > 0$. *Suppose for simplicity that S is obtained from its minimal model* $S_{\min}$ *by blowing up distinct points. Let* $E_1, \ldots, E_r$ *be the exceptional curves in S. Suppose that* $S \cong A \# B$ *is an orientation-preserving diffeomorphism with the intersection form of B being negative definite. Then* $H^2(B) \subset H^2(S)$ *is contained in the subspace spanned by the* $[E_i]$, $1 \leq i \leq r$. *In particular, if S is minimal, then B is a homotopy 4-sphere.*

The following is an immediate consequence.

Corollary 7. *Let S and S′ be algebraic surfaces with* $p_g(S) > 0$. *Suppose that both S and S′ are blow ups of either simply connected elliptic surfaces or complete intersections. Suppose for simplicity that S and S′ are each obtained from their minimal models by blowing up distinct points. Let* $E_1, \ldots, E_r$ *be the exceptional curves in S, and let* $E'_1, \ldots E'_{r'}$ *be the exceptional curves in S′. Let* $f: S \to S'$ *be an orientation-preserving diffeomorphism. Then*

$$f^*\left(\bigoplus_i \mathbb{Z}([E_i])\right) = \bigoplus_i \mathbb{Z}([E'_i]).$$

In particular, $r = r'$.

Here is another consequence.

Corollary 8. *Let S be an algebraic surface. Suppose that S is orientation-preserving diffeomorphic to an r-fold blow up of a minimal, simply connected elliptic surface* $S'_{\min}$ *with* $p_g(S'_{\min}) > 0$ *and with multiple fibers of multiplicities* p'_1 *and* p'_2. *Then S is an r-fold* *blow up of a minimal, simply connected elliptic surface $S_{\min}$ with $p_g(S_{\min}) = p_g(S'_{\min})$. Moreover, if the multiple fibers of $S_{\min}$ have multiplicities p_1 and p_2, then $p_1 p_2 = p'_1 p'_2$.*

To prove this result one notices that, by Theorem 6, if S is an algebraic surface which is orientation-preserving diffeomorphic to an r-fold blow up of a minimal simply connected elliptic surface, then S itself is obtained from its minimal model $S_{\min}$ by at most r blow ups. Hence $K^2_{S_{\min}} \leq 0$, and thus $S_{\min}$ is not a surface of general type. But a minimal surface, not of general type, which is diffeomorphic to a blown up simply connected elliptic surface is itself an elliptic surface or a $K3$ surface. Hence, $S_{\min}$ deforms to an elliptic surface. We now invoke Theorem 2.

In particular, every complex surface diffeomorphic to a $K3$ surface is again a $K3$ surface, answering a question implicitly raised by Kodaira [10].

Remark 9. (a) It is natural to ask if the diffeomorphism type of a simply connected non-rational elliptic surface determines the pair of multiplicities of the multiple fibres, rather than just the product. It is quite possible that a more detailed analysis of the relevant moduli spaces and invariants will give this result.

(b) There are natural generalizations of our results to elliptic surfaces with finite fundamental group which follow by considering the universal covering, which is a simply connected elliptic surface. These results generalize results of Lübke-Okonek [11] and Maier [12].

References

[1] S. Donaldson, Anti-self-dual Yang-Mills connections over a complex algebraic surface and stable vector bundles, Proc. London Math. Soc. (3) 50 (1985) 1-26.

[2] S. Donaldson, Irrationality and the h-cobordism conjecture, J. Differential Geom. 26 (1987) 141-168.

[3] S. Donaldson, Polynomial invariants for smooth four-manifolds, to appear.

[4] M. Freedman, The topology of four-dimensional manifolds, J. Differential Geom. 17 (1982) 357-454.

[5] R. Friedman, Stable rank 2 bundles on regular elliptic surfaces, Invent. Math., to appear.

[6] R. Friedman, B. Moishezon and J. Morgan, On the C^∞-invariance of the canonical classes of certain algebraic surfaces, Bull. Amer. Math. Soc. (N.S.) 17 (1987) 283-286.

[7] R. Friedman and J. Morgan, On the diffeomorphism type of certain algebraic surfaces I, J. Differential Geom. 27 (1988) 297-369.

[8] R. Friedman and J. Morgan, On the diffeomorphism type of certain algebraic surfaces II, J. Differential Geom., 27 (1988) 371-398.

[9] R. Friedman and J. Morgan, Algebraic surfaces and 4-manifolds: some conjectures and speculations, Bull. Amer. Math. Soc. (N.S.) 18 (1988) 1-19.

[10] K. Kodaira, On homotopy $K3$ surfaces, in: Essays on Topology and Related Topics, Mémoires Dédiés á Georges deRham (Springer, Berlin, 1970) 58-69.

[11] M. Lübke and C. Okonek, Differentiable structures on elliptic surfaces with finite fundamental group, Compositio Math. 63 (1987) 217-222.

[12] F. Maier, Tulane Thesis (1987).